4a.2 Exact differential equations

If a function $y = y(x)$ is given implictly through

$$F(x, y(x)) = C \ , \ C \in \mathbb{R} \ ,$$

it solves the differential equation

$$0 = \frac{\mathrm{d}F(x, y(x))}{\mathrm{d}x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} \ .$$

Example: Concentric circles are described by

$$x^2 + y^2 = r^2 \ , \ r > 0 \ ,$$

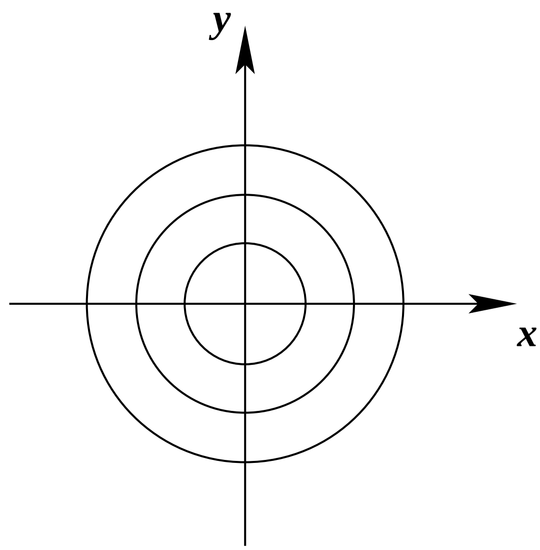
which leads to the ODE

$$2x + 2y\,y' = 0 \ .$$

In explicit form:

$$y' = -\frac{x}{y} \ .$$

Problem: $y' \to \pm\infty$ at $(x, y) = (r, 0)$.



The use of parametric curves $(x(t), y(t))$ instead of functions $y = y(x)$ avoids this problem:

$$F(x(t), y(t)) \; = \; C \;, \; C \in \mathbb{R} \;,$$

gives

$$0 \; = \; \frac{\mathsf{d}F(x(t), y(t))}{\mathsf{d}t} \; = \; \frac{\partial F}{\partial x}\frac{\mathsf{d}x}{\mathsf{d}t} + \frac{\partial F}{\partial y}\frac{\mathsf{d}y}{\mathsf{d}t}$$

or

$$0 \; = \; \mathsf{d}F \; = \; \frac{\partial F}{\partial x}\mathsf{d}x + \frac{\partial F}{\partial y}\mathsf{d}y \; .$$

Question:

Which condition must two functions $p(x,y)$ and $q(x,y)$ satisfy if the ODE

$$p(x,y)\,\dot{x} + q(x,y)\,\dot{y} = 0$$

is supposed to be the total derivative of a function $F = F(x(t), y(t))$ with respect to t?

Recall multivariable calculus:

Since

$$\frac{\partial F}{\partial x} \stackrel{!}{=} p\ , \ \frac{\partial F}{\partial y} \stackrel{!}{=} q\ , \quad \text{and} \quad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

(for sufficiently smooth F), the condition

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$$

must be fulfilled (and is sufficient if the definition domain of p and q satisfy certain topological constraints).

This leads to the following **definition**:
If the condition

$$\boxed{\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}}$$

is fulfilled, then the differential equation

$$\boxed{p(x,y)\,\dot{x} + q(x,y)\,\dot{y} = 0 \qquad (*)}$$

is called **exact**. Solutions $(x(t), y(t))$ are called **integral curves**.

Existence and uniqueness of solutions:
Suppose the functions p and q are continuous, $p^2 + q^2 > 0$ in their definition domain, and the ODE $(*)$ is exact. Then all integral curves can be written as

$$\boxed{F(x,y) = C \ , \ C \in \mathbb{R} \ ,}$$

where

$$\boxed{\frac{\partial F}{\partial x} = p \quad \text{and} \quad \frac{\partial F}{\partial y} = q \ .}$$

Such a function F ('antiderivative') can be obtained by means of **line integration** of the vector-valued function of two variables

$$\vec{V} = \begin{pmatrix} p \\ q \end{pmatrix}$$

along any curve $\mathcal{C}(x,y)$ from an arbitrary fixed point (x_*, y_*) to (x,y).

If $\mathcal{C}(x,y)$ is parametrised by $\vec{r} : [a,b] \to I\!R^2$,

$$\vec{r}(t) = \begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \end{pmatrix} ,$$

then

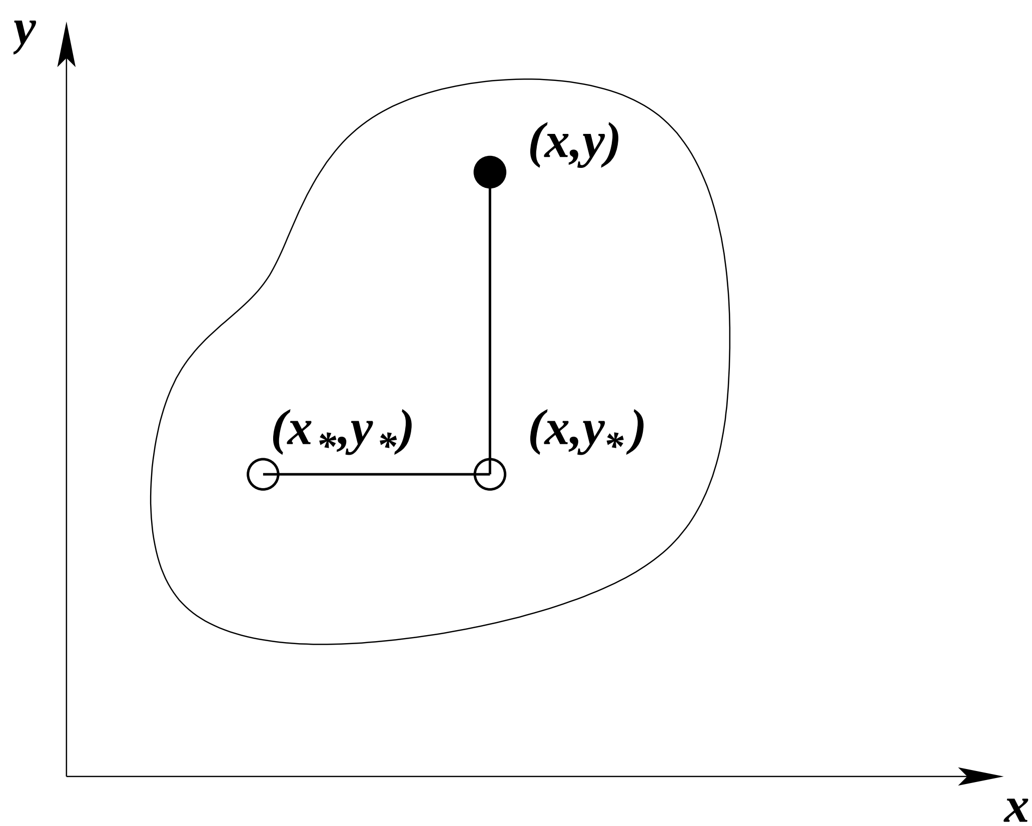
$$\begin{aligned} F(x,y) &= \int_{\mathcal{C}(x,y)} \vec{V} \cdot \mathrm{d}\vec{r} \\ &= \int_{\mathcal{C}(x,y)} \begin{pmatrix} p(\tilde{x},\tilde{y}) \\ q(\tilde{x},\tilde{y}) \end{pmatrix} \cdot \mathrm{d} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \\ &= \int_{\mathcal{C}(x,y)} [p(\tilde{x},\tilde{y})\,\mathrm{d}\tilde{x} + q(\tilde{x},\tilde{y})\,\mathrm{d}\tilde{y}] \\ &= \int_a^b \left[p(\tilde{x},\tilde{y}) \frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} + q(\tilde{x},\tilde{y}) \frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} \right] \mathrm{d}t \ . \end{aligned}$$

If $\mathcal{C}(x, y)$ can be chosen to consist of two straight line segments $\mathcal{C}_1$ and $\mathcal{C}_2$ which are parallel to the x-axis and the y-axis, respectively, that means,

$$\mathcal{C}_1 : (x_*, y_*) \to (x, y_*) \ , \quad \mathcal{C}_2 : (x, y_*) \to (x, y) \ ,$$

and which are traversed with unit speed, then

$$\begin{aligned} F(x, y) \ &= \ \int_{x_*}^{x} p(\tilde{x}, \tilde{y} = y_*) \, \mathrm{d}\tilde{x} \\ &\quad + \int_{y_*}^{y} q(\tilde{x} = x, \tilde{y}) \, \mathrm{d}\tilde{y} \ . \end{aligned}$$



Example: The ODE

$$(2x + y^2)\,\dot{x} \,+\, 2xy\,\dot{y} \;=\; 0$$

is exact:

$$\frac{\partial(2x + y^2)}{\partial y} \;=\; 2y \;=\; \frac{\partial(2xy)}{\partial x} \;.$$

We may choose $(x_*, y_*) = (0, 0)$ and get

$$\int_0^x p(\tilde{x}, \tilde{y} = 0)\,\mathrm{d}\tilde{x} \;\;=\;\; \int_0^x 2\tilde{x}\,\mathrm{d}\tilde{x} \;=\; x^2 \;,$$
$$\int_0^y q(\tilde{x} = x, \tilde{y})\,\mathrm{d}\tilde{y} \;\;=\;\; \int_0^y 2x\tilde{y}\,\mathrm{d}\tilde{y} \;=\; xy^2 \;,$$

which yields

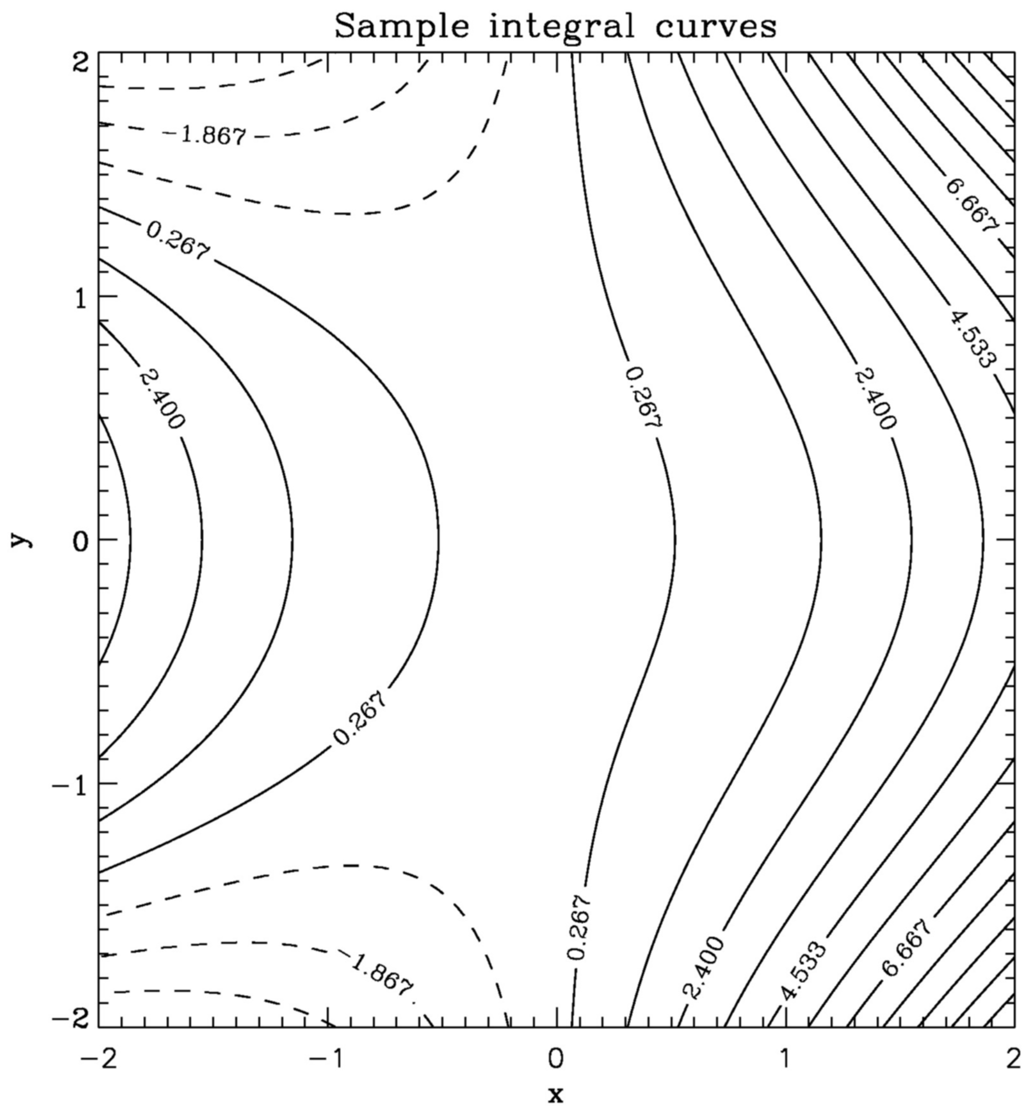
$$F(x, y) \;=\; x^2 + xy^2 \;=\; x(x + y^2) \;.$$

All solutions of the ODE are thus given by

$$x(x + y^2) \;=\; C \;,\; C \in I\!R \;.$$

Integral curves of exact differential equations are graphically represented as **contour plots**.

Example: Contour lines of $F(x, y) = x(x+y^2)$.



Integrating factors

Question: What if the differential equation

$$p(x,y)\,\dot{x} + q(x,y)\,\dot{y} = 0$$

is **not exact**?

Answer: Find a function $\phi(x,y)$ such that

$$\phi(x,y)\,p(x,y)\,\dot{x} + \phi(x,y)\,q(x,y)\,\dot{y} = 0$$

is exact, that means,

$$\frac{\partial(\phi p)}{\partial y} = \frac{\partial(\phi q)}{\partial x}\ .$$

Then ϕ is called an **integrating factor**.

It can be shown that such a function $\phi = \phi(x,y)$ always exists.

In practice, however, it may be difficult to actually find an integrating factor.

Integrating factors (cont.)

Ansatz: ϕ depends only on one variable, e.g.,

$$\phi \;=\; \phi(x) \;.$$

Then

$$\phi(x)\,\frac{\partial p}{\partial y} \;=\; \phi'(x)\,q + \phi(x)\,\frac{\partial q}{\partial x}$$

which leads to

$$\frac{1}{q}\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) \;=\; \frac{\phi'(x)}{\phi(x)} \;=\; (\ln\phi)'(x) \;.$$

An integrating factor which only depends on x can only exist if the left-hand side does not depend on y.

If $\phi = \phi(y)$, then

$$\frac{1}{p}\left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) \;=\; \frac{\phi'(y)}{\phi(y)} \;=\; (\ln\phi)'(y) \;,$$

and the left-hand side must not depend on x.

Integrating factors (cont.)

Example: The ODE

$$(y^2 + x + 1)\,\dot{x} + 2y\,\dot{y} \;=\; 0$$

is not exact.

Since

$$\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} \;=\; \frac{\partial}{\partial y}(y^2 + x + 1) - \frac{\partial}{\partial x}(2y) \;=\; 2y \;,$$

the function

$$\frac{1}{q}\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) \;=\; \frac{2y}{2y} \;=\; 1$$

does not depend on y and we can find an integrating factor of the form $\phi = \phi(x)$ as a solution of

$$(\ln\phi)'(x) \;=\; 1 \;,$$

for example,

$$\phi(x) \;=\; \mathrm{e}^x \;.$$

Integrating factors (cont.)

Thus the differential equation

$$(y^2 + x + 1)\mathrm{e}^x\,\dot{x} + 2y\mathrm{e}^x\,\dot{y} = 0$$

is exact.

The 'antiderivative' $F = F(x, y)$ could again be determined by line integration. As an alternative, integrate

$$\frac{\partial F}{\partial y} = q(x, y) = 2y\mathrm{e}^x$$

with respect to y to get

$$F(x, y) = y^2\mathrm{e}^x + h(x) \ .$$

Differentiating with respect to x gives

$$\frac{\partial F}{\partial x} = y^2\mathrm{e}^x + h'(x)$$

which must be compared with

$$p(x, y) = y^2\mathrm{e}^x + (x + 1)\mathrm{e}^x \ .$$

Integrating factors (cont.)

Therefore,

$$h'(x) \;=\; (x+1)\mathrm{e}^{x} \;.$$

This is solved by, e.g.,

$$h(x) \;=\; x\mathrm{e}^{x}$$

which leads to

$$F(x,y) \;=\; (y^2 + x)\,\mathrm{e}^{x} \;.$$

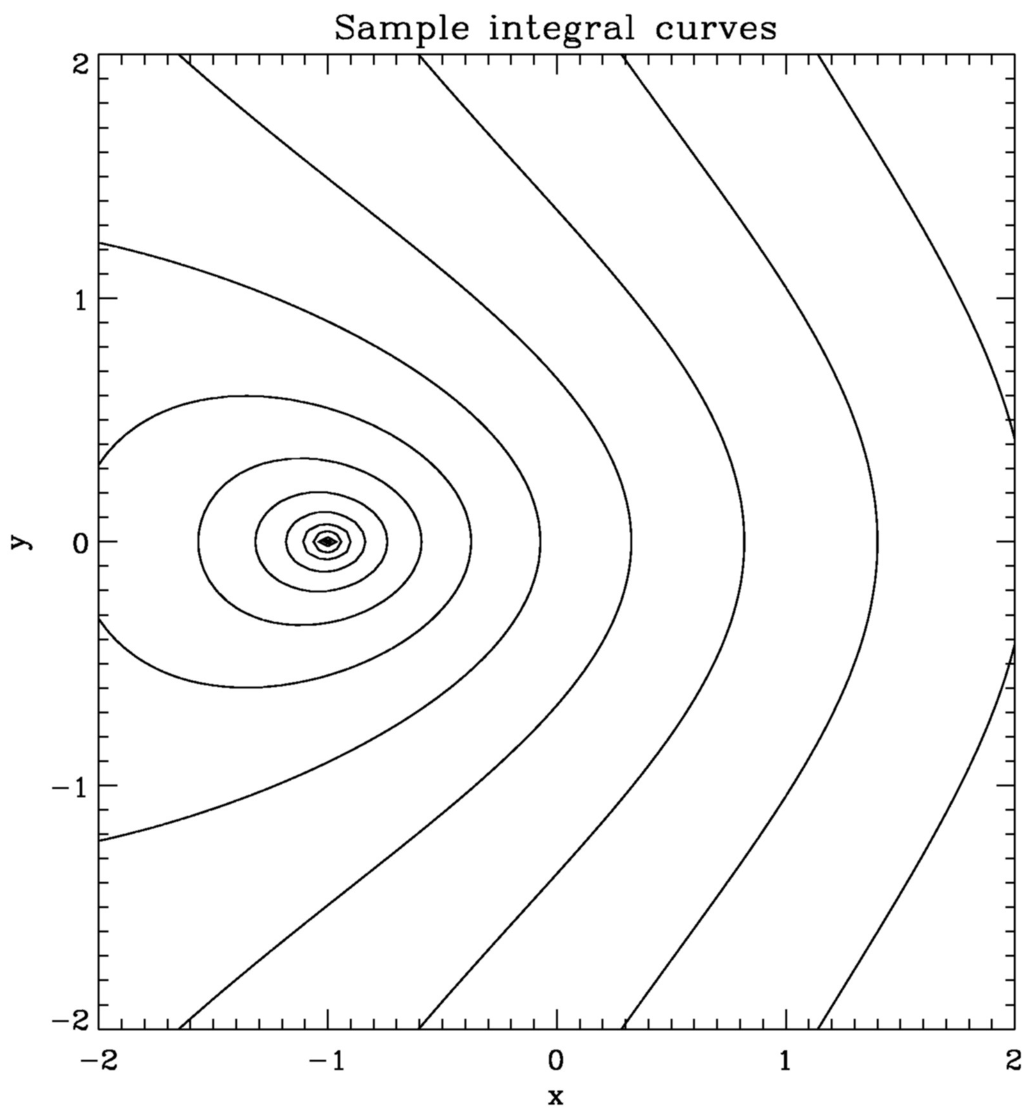
Conclusion: The solutions of

$$(y^2 + x + 1)\,\dot{x} \;+\; 2y\,\dot{y} \;=\; 0$$

can be written as

$$F(x,y) \;=\; (y^2 + x)\,\mathrm{e}^{x} \;=\; C \;,\; C \in \mathbb{R} \;.$$

Contour lines of $F(x, y) = (y^2 + x)\,\mathrm{e}^x$:



Note that at $(x, y) = (-1, 0)$ both $p(x, y)$ and $q(x, y)$ are zero.