AN EXTENSION OF THE PIATETSKI-SHAPIRO PRIME NUMBER THEOREM

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Abstract:

Balog and Harman proved that for any λ in the interval $1/2 \le \lambda < 1$ and any real θ there are infinitely many primes p satisfying $\left\{p^\lambda - \theta\right\} < p^{-(1-\lambda)/2+\varepsilon}$ (with an asymptotic result). In the present paper we prove that for $59/85 = 0.694... < \lambda < 1$ the above exponent $-(1-\lambda)/2+\varepsilon$ may be replaced by $-\min\{\max\{(35-22\lambda)/129, 1/7\}, 5/18-\lambda/6\}+\varepsilon$. This result in particular contains the Piatetski-Shapiro prime number theorem in the version given by Liu and Rivat: We have $|\{n \le N : [n^c] \text{ prime}\}| \sim N/(c\log N)$ as $N \to \infty$ if $1 < c < 15/13$. For the proof of our result we use exponential sum techniques.

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1 Introduction

In [2], [3], [4], [5], [6], [8] and [9], Balog, Harman and the author of the present paper studied the distribution of fractional parts of p^λ, p running over the prime numbers and λ being a given real number lying in the interval $0 < \lambda < 1$. Balog's and Harman's result for the range $1/2 \le \lambda < 1$ (see [4], [8] and [6, Theorem 2]) can be formulated as follows.

THEOREM 1: *Suppose that $B > 0$, $\lambda \in [1/2, 1)$ and a real θ are given. Then for any fixed τ in the range $0 \le \tau < (1-\lambda)/2$ we have*

$$\sum_{\substack{n \le N,\\ \left\{n^\lambda - \theta\right\} < n^{-\tau}}} \Lambda(n) = \frac{N^{1-\tau}}{1-\tau} + O\left(\frac{N^{1-\tau}}{(\log N)^B}\right) \tag{1}$$

as $N \to \infty$.

Here, as in the following, $\Lambda(n)$ denotes the von Mangoldt function.

In particular, THEOREM 1 implies that $\left\{p^{1/2}\right\} < p^{-1/4+\varepsilon}$ for infinitely many primes p. Recently, Harman and Lewis [10] proved that the above exponent $-1/4+\varepsilon$ may be replaced by -0.262. However, their method works only for $\lambda = 1/2$. Already about fifty years ago Ankeny [1] and Kubilius [13] proved that, provided the Riemann Hypothesis for Hecke L-series with Größencharacters over $\mathbb{Q}(i)$ holds true, $\left\{p^{1/2}\right\} < p^{-1/2+\varepsilon}$ for infinitely many primes p. If the term ε in the latter exponent $-1/2+\varepsilon$ could be removed, we knew that n^2+1 is prime infinitely many often. However, it seems to be extremely difficult to prove this - even conditionally.

It is known that THEOREM 1 may be improved for λ close to 1 (see [4]) using exponential sum techniques like in the proof of the Piatetski-Shapiro prime number theorem. However, as far as the author knows, this has never been carried out explicitly in the literature. The purpose of the present paper is to fill this gap. We will improve on THEOREM 1 in the range $59/85 = 0.694... < \lambda < 1$. Our main result is

THEOREM 2: *Suppose that $B > 0$, $\lambda \in (59/85, 1)$ and a real θ are given. Define*

$$G(\lambda) := \min\{\max\{(35-22\lambda)/129, 1/7\}, 5/18-\lambda/6\}$$

$$= \begin{cases} (35-22\lambda)/129 & \text{if } 59/85 < \lambda \le 58/77, \\ 1/7 & \text{if } 58/77 < \lambda \le 17/21, \\ 5/18-\lambda/6 & \text{if } 17/21 < \lambda < 1. \end{cases}$$

Then for any fixed τ in the range $0 \le \tau < G(\lambda)$ we have (1) as $N \to \infty$.

THEOREM 2 may be viewed as an extension of the Piatetski-Shapiro prime number theorem in the version given by Liu and Rivat [14]:

THEOREM 3: *Suppose that $0 \le \theta < 1$. Then for $1 < c < 15/13 = 1.1538...$ we have*

$$|\{n \le x \ : \ [(n+\theta)^c] \text{ prime}\}| \sim \frac{x}{c \log x} \tag{2}$$

as $x \to \infty$.

The first result of this kind was obtained by Piatetski-Shapiro [15] who showed that (2) with $\theta = 0$ holds true for $1 < c < 12/11$. The c-range has been improved by many authors. The latest record is $1 < c < 2817/2426 = 1.161...$ obtained by Rivat and Sargos [16].

THEOREM 3 can be easily deduced from THEOREM 2 by putting $\lambda := 1/c$ and $\tau := 1-\lambda$. We point out that $1-\lambda < G(\lambda)$ if $13/15 < \lambda < 1$.

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2 Reduction to exponential sums

The standard procedure for attacking problems related to the Piatetski-Shapiro prime number theorem (see [11], [12], [14], [15], [16], for instance) is as follows: Firstly, one reduces the sum of $\Lambda(n)$ in question (in our paper, this is the sum on the left side of (1)) to exponential sums of the form

$$\sum_{n\sim N} \Lambda(n) e\left(hn^{\lambda}\right).$$

Secondly, one breaks up these exponential sums into certain bilinear exponential sums (so-called type I and type II sums) by using Vaughan-type identities. Finally, one estimates these bilinear exponential sums in a non-trivial way. Here we also follow this approach. However, since the first two steps were completely carried out in several papers (see [11] for example), we omit some details in these steps.

Throughout the following, we suppose that λ, τ and t are given real numbers satisfying $1/2 \le \lambda < 1$ and $0 \le \tau < t < 1$, and we put

$$H := N^t.$$

The parameter t will later be choosen suitably in dependence on λ and τ. Furthermore, we suppose that η is a (small) positive real number. By the notation $k \sim K$ we mean k to be lying in some interval $K_1 \le k \le K_2$ with $K/2 \le K_1 \le K_2 \le 2K$.

Using the Fourier series expansion of $\{x\}$, we obtain

LEMMA 1: *Suppose that* (a_h) *is a sequence of complex numbers. Assume that we uniformly have, as* $N \to \infty$,

$$\sum_{1\le h\le H} a_h \sum_{N<n\le 2N} \Lambda(n)\left(1-e\left(hn^{-\tau}\right)\right) e\left(h\left(n^{\lambda}-\theta\right)\right) = O\left(N^{1-\tau-\eta}\right) \tag{3}$$

if $|a_h| \le 1/h$ $(h \in \mathbb{N})$. *Then the asymptotic estimate* (1) *holds true.*

From LEMMA 1, using partial summation and the bounds

$$1-e\left(hn^{-\tau}\right) \ll hn^{-\tau}, \tag{4}$$

$$\frac{\partial}{\partial x}\left(1-e\left(hx^{-\tau}\right)\right) \ll hx^{-\tau-1}, \tag{5}$$

we obtain

LEMMA 2: *We have* (1) *if*

$$\sup_{N<N_1\le 2N} \sum_{1\le h\le H} \left| \sum_{N<n\le N_1} \Lambda(n) e\left(hn^{\lambda}\right)\right| = O\left(N^{1-\eta}\right) \tag{6}$$

for a suitable $\eta > 0$.

Employing an identity given in [11] for $\Lambda(n)$, we obtain

<u>Lemma 3:</u> *Suppose that u, v and z are real parameters satisfying the conditions*

$$3 \le u < v < z < 2N, \;\; z - 1/2 \in \mathbb{N}, \;\; z \ge 4u^2, \;\; N \ge 32z^2u, \;\; v^3 \ge 64N.$$

Suppose further that $1 \le Y \le N$, $XY = N$ and $1 \le J \le H$. Let (a_m), (b_n), (c_h) be sequences of complex numbers. We write

$$K := \sum_{m \sim X} \sum_{\substack{n \sim Y, \\ mn \sim N}} \sum_{h \sim J} a_m c_h e\left(hm^\lambda n^\lambda\right)$$

and

$$L := \sum_{m \sim X} \sum_{\substack{n \sim Y, \\ mn \sim N}} \sum_{h \sim J} a_m b_n c_h e\left(hm^\lambda n^\lambda\right).$$

Then the estimate (6) *holds true if we uniformly have*

$$K \ll N^{1-2\eta} \quad \text{for } Y \ge z, \;\; J \le H \text{ and } |a_m|, |c_h| \le 1 \;\, (m, h \in \mathbb{N})$$

and

$$L \ll N^{1-2\eta} \quad \text{for } u \le Y \le v, \;\; J \le H \text{ and } |a_m|, |b_n|, |c_h| \le 1 \;\, (m, n, h \in \mathbb{N}).$$

3 Treatment of L

From now on, we suppose that $1 \le Y \le N$, $XY = N$, $1 \le J \le H$, and that (a_m), (b_n), (c_h) are sequences of complex numbers such that $|a_m| \le 1$, $|b_n| \le 1$, $|c_h| \le 1$ for all $m, n, h \in \mathbb{N}$.

The following lemma has been established by Heath-Brown (see [11], page 257).

<u>Lemma 4:</u> *Let Q be any positive integer and ε be any positive real number. Then*

$$|L|^2 \ll \left(QX\left(JX^\lambda Y^\lambda Q^{-1}\right)^{1/2}\left(J^2Q^{-1}Y^2 + JY\right) + QX^{2-\lambda}\left(JY^{2-\lambda} + JYX^\lambda\right)\right)N^\varepsilon.$$

To optimize the estimate in Lemma 4, we choose

$$Q := 1 + \left[JX^{(\lambda-2)/3}Y^{(\lambda+2)/3}\right].$$

Then, by a short calculation using $J \le H = N^t$ and $XY = N$, we obtain

LEMMA 5: *We have*

$$|L|^2 \ll \Big(N^{1+\lambda/2+3t/2} + N^{2-\lambda+t} + N^{2+t}Y^{-1} + N^{4/3+\lambda/3+2t}Y^{1/3} + N^{2/3+2\lambda/3+2t}Y^{2/3} + N^{4/3-2\lambda/3+2t}Y^{4/3}\Big)\, N^{\varepsilon}.$$

This implies

LEMMA 6: *For every sufficiently small fixed $\eta > 0$ we have*

$$L \ll N^{1-2\eta}$$

when

$$N^{t+100\eta} \le Y \le N^{2-\lambda-6t-100\eta},$$

provided the conditions

$$t < \lambda, \quad t < (2-\lambda)/3, \quad t > (1-\lambda)/3 \tag{7}$$

are satisfied.

4 Treatment of K

Heath-Brown established the following two estimates for the term

$$K_h := \sum_{m \sim X} \sum_{\substack{n \sim Y, \\ mn \sim N}} a_m e\left(hm^{\lambda}n^{\lambda}\right)$$

(see [11], pages 261-262).

LEMMA 7: *(i) We have*

$$K_h \ll (\log N)^2 \left(N^{1-\lambda}h^{-1} + Xh^{1/2}N^{\lambda/2}\right).$$

(ii) Let (p,q) be any exponent pair for which $0 < p \le 1/2 \le q \le 1$. Let P be any positive integer. Then

$$|K_h|^2 \ll (\log N)^8 (X + NP^{-1}) \Big(N + N^{(p+1/2)\lambda}Y^{q-2p-1/2}H^{p+1/2}P^{p+3/2} + N^{1-\lambda/2}Y^{1/2}h^{-1/2}P^{1/2} + YP\Big).$$

Since

$$|K| \le \sum_{J/2 \le h \le 2J} |K_h|, \tag{8}$$

from the estimates in Lemma 7, we can immediately deduce two estimates for K. We take the exponent pair $(p,q)=(2/7,4/7)$, assume that $t<14/11-\lambda$, and choose

$$P := \left[N^{(2-(1+2p)(\lambda+t))/(2p+3)}Y^{(4p+1-2q)/(2p+3)}\right] = \left[N^{(14-11(\lambda+t))/25}Y^{7/25}\right]. \tag{9}$$

Then, from Lemma 7 and (8), we obtain

Lemma 8: *(i) We have*

$$K \ll (\log N)^3\left(N^{1-\lambda}+XN^{(\lambda+3t)/2}\right).$$

(ii) We have

$$K \ll (\log N)^4\left(N^{(36+11\lambda+61t)/50}Y^{-7/50}+N^{(43-7\lambda+43t)/50}Y^{9/50}+N^{1/2+t}Y^{1/2}\right),$$

provided the conditions

$$t<14/11-\lambda, \qquad Y\ge N^{(14-11(\lambda+t))/18} \tag{10}$$

are satisfied.

Alternatively, adapting the method in [14], we can employ the following estimate of Fouvry-Iwaniec [7, Theorem 3] for trilinear exponential sums with monomials to estimate the complete triple sum K, taking advantage of the effect arising from the summation over h.

Lemma 9: *Let α, α_1, α_2 be real constants such that $\alpha\neq 1$, $\alpha\alpha_1\alpha_2\neq 0$. Let M, M_1, M_2, $x\ge 1$ and $|\phi_m|\le 1$, $|\psi_{m_1,m_2}|\le 1$. We then have*

$$\begin{aligned}&\sum_{m\sim M}\sum_{m_1\sim M_1}\sum_{m_2\sim M_2}\phi_m\psi_{m_1,m_2}e\left(x\frac{m^\alpha m_1^{\alpha_1}m_2^{\alpha_2}}{M^\alpha M_1^{\alpha_1}M_2^{\alpha_2}}\right)\\ \ll\ &\left(x^{1/4}M^{1/2}(M_1M_2)^{3/4}+M^{7/10}M_1M_2+M(M_1M_2)^{3/4}+x^{-1/4}M^{11/10}M_1M_2\right)\\ &(\log(2MM_1M_2))^2.\end{aligned}$$

To handle K, we firstly separate the dependence over the range of summation of m and n by a variant of Perron's formula (see [7, Lemma 6], for instance). In this manner, we end up with trilinear sums of the form

$$\sum_{m\sim X}\sum_{n\sim Y}\sum_{h\sim J}\phi_m\psi_n\epsilon_h e\left(hm^\lambda n^\lambda\right).$$

We then estimate these sums by applying Lemma 9 with the following choice of parameters:

$$x:=JN^\lambda, \quad M:=Y, \quad M_1:=X, \quad M_2:=J, \quad \alpha:=\lambda, \quad \alpha_1:=\lambda, \quad \alpha_2:=1.$$

Additionally taking $J \leq H = N^t$ and $XY = N$ into account, we arrive at the following estimate for K.

LEMMA 10: *We have*

$$K \ll \left(N^{3/4+\lambda/4+t}Y^{-1/4} + N^{1+t}Y^{-3/10} + N^{3/4+3t/4}Y^{1/4} + N^{1-\lambda/4+3t/4}Y^{1/10}\right)(\log N)^3.$$

Another possible choice of the parameters in LEMMA 9 is:

$$x := JN^{\lambda}, \quad M := X, \quad M_1 := Y, \quad M_2 := J, \quad \alpha := \lambda, \quad \alpha_1 := \lambda, \quad \alpha_2 := 1.$$

This choice leads to the following estimate for K.

LEMMA 11: *We have*

$$K \ll \left(N^{1/2+\lambda/4+t}Y^{1/4} + N^{7/10+t}Y^{3/10} + N^{1+3t/4}Y^{-1/4} + N^{11/10-\lambda/4+3t/4}Y^{-1/10}\right)(\log N)^3.$$

Next, we successively process the estimates given in LEMMAS 8, 10 and 11 to determine on which conditions we have $K \ll N^{1-2\eta}$. From LEMMA 8, we obtain

LEMMA 12: (*i*) *For every sufficiently small fixed* $\eta > 0$ *we have*

$$K \ll N^{1-2\eta} \tag{11}$$

when

$$N^{(\lambda+3t)/2+100\eta} \leq Y.$$

(*ii*) *For every sufficiently small fixed* $\eta > 0$ *we have* (11) *when*

$$N^{(11\lambda+61t)/7-2+100\eta} \leq Y \leq N^{\min\{1-2t,(7(1+\lambda)-43t)/9\}-100\eta},$$

provided the conditions

$$t < 14/11 - \lambda, \qquad t > (14-11\lambda)/47 \tag{12}$$

are satisfied.

From LEMMAS 10 and 11, we derive

LEMMA 13: *For every sufficiently small fixed* $\eta > 0$ *we have* (11) *when*

$$N^{\max\{10t/3,\lambda+4t-1\}+100\eta} \leq Y \leq N^{1-3t-100\eta}$$

or

$$N^{3t+100\eta} \leq Y \leq N^{\min\{1-10t/3,2-\lambda-4t\}-100\eta},$$

provided the condition

$$t < (5\lambda - 2)/9 \tag{13}$$

is satisfied.

Combining LEMMAS 12 and 13, we obtain

LEMMA 14: (i) *For every sufficiently small fixed* $\eta > 0$ *we have* (11) *when*

$$N^{(11\lambda+61t)/7-2+100\eta} \leq Y,$$

provided the condition (12) *as well as the conditions*

$$t < (2-\lambda)/7, \quad t < (14+5\lambda)/113 \tag{14}$$

are satisfied.

(ii) *For every sufficiently small fixed* $\eta > 0$ *we have* (11) *when*

$$N^{3t+100\eta} \leq Y,$$

provided the conditions in (12), (13), (14) *as well as the conditions*

$$t < (21-11\lambda)/82, \quad t < 3/20, \quad t < (3-2\lambda)/8 \tag{15}$$

are satisfied.

PROOF: To prove part (i), it suffices to show that

$$(\lambda + 3t)/2 < \min\{1-2t, (7(1+\lambda) - 43t)/9\}.$$

This actually follows from the conditions in (14).

To prove part (ii), we additionally have to show that

$$(11\lambda + 61t)/7 - 2 < 1 - 3t \tag{16}$$

and

$$\max\{10t/3, \lambda + 4t - 1\} < \min\{1 - 10t/3, 2 - \lambda - 4t\}. \tag{17}$$

The inequality (16) follows from the first condition in (15), and (17) follows from the second and the third condition in (15).

5 Proof of Theorem 2

Firstly, we suppose that $58/77 < \lambda < 1$ and

$$0 \leq \tau < \min\{1/7, 5/18 - \lambda/6\}.$$

We choose t in such a manner that

$$\max\{\tau, (1-\lambda)/3, (14-11\lambda)/47, 1/9\} < t < \min\{1/7, 5/18-\lambda/6\},$$

and we take

$$\begin{aligned} u &:= N^{t+100\eta}, \\ v &:= 4N^{1/3}, \\ z &:= \left[N^{3t+100\eta}\right] + 1/2. \end{aligned}$$

Then the conditions in (7), (12), (13), (14), (15) to λ and t are satisfied. To see that the conditions in LEMMA 3 to u, v, z are also satsified, we notice that, if η is sufficiently small and N is sufficiently large, we have $v < z$ because of $t > 1/9$, and $N \geq 32z^2u$ because of $t < 1/7$.

Now, combining LEMMA 2, LEMMA 3, LEMMA 6 and LEMMA 14(ii), and taking $1/3 < 2-\lambda-6t$ by $t < 5/18-\lambda/6$ into consideration, we obtain the result of THEOREM 2 in the range $58/77 < \lambda < 1$.

Secondly, we suppose that $59/85 < \lambda \leq 58/77$ and

$$0 \leq \tau < (35-22\lambda)/129.$$

We choose t in such a manner that

$$\max\{\tau, (1-\lambda)/3, (14-11\lambda)/47, (49-33\lambda)/183\} < t < (35-22\lambda)/129,$$

and we take

$$\begin{aligned} u &:= N^{t+100\eta}, \\ v &:= 4N^{1/3}, \\ z &:= \left[N^{(11\lambda+61t)/7-2+100\eta}\right] + 1/2. \end{aligned}$$

Then the conditions in (7), (12), (14) to λ and t are satisfied. To see that the conditions in LEMMA 3 to u, v, z are also satsified, we notice that, if η is sufficiently small and N is sufficiently large, we have $v < z$ because of $t > (49-33\lambda)/183$, $z \geq 4u^2$ because of $t > (14-11\lambda)/47$, and $N \geq 32z^2u$ because of $t < (35-22\lambda)/129$.

Now, combining LEMMA 2, LEMMA 3, LEMMA 6 and LEMMA 14(i), we obtain the result of THEOREM 2 in the range $59/85 < \lambda < 58/77$. This completes the proof. □

References

[1] N.C. Ankeny, *Representations of primes by quadratic forms*, American J. Math. 74 (1952), 913-919.

[2] S. Baier, *On the p^λ problem*, to appear in Acta Arithmetica.

[3] S. Baier, *Estimates for sums over zeta zeros of Dirichlet polynomials, and an application*, preprint.

[4] A. Balog, *On the fractional part of p^θ*, Arch. Math. (Basel) 40 (1983), 434-440.

[5] A. Balog, *On the distribution of p^θ mod* 1, Acta Math. Hung. 45 (1985), 179-199.

[6] A. Balog, G. Harman, *On mean values of Dirichlet polynomials*, Arch. Math. (Basel) 57 (1991), 581-587.

[7] E. Fouvry, H. Iwaniec, *Exponential sums with monomials*, J. Number Theory 33 (1989), 311-333.

[8] G. Harman, *On the distribution of $\sqrt{p}$ modulo one*, Mathematika 30 (1983), 104-116.

[9] G. Harman, *Fractional and integral parts of p^λ*, Acta Arith. 58 (1991), 141-152.

[10] G. Harman, P. Lewis, *Gaussian primes in narrow sectors*, Mathematika 48 (2001), 119-135.

[11] D.R. Heath-Brown, *The Pjateckii-Shapiro Prime Number Theorem*, J. Number Theory 16 (1983), 242-266.

[12] G.A. Kolesnik, *The distribution of primes in sequences of the form* $[n^c]$, Mat. Zametki 2 (2) (1972), 117-128.

[13] J.P. Kubilius, *On a problem in the n-dimensional analytic theory of numbers*, Viliniaus Valst. Univ. Mokslo dardai Chem. Moksly, Ser. 4 (1955), 5-43.

[14] H.Q. Liu, J. Rivat, *On the Pjateckii-Shapiro Prime Number Theorem*, Bull. London Math. Soc. 24 (1992), 143-147.

[15] I.I. Piatetski-Shapiro, *On the distribution of prime numbers in the sequence of the form* $[f(n)]$, Mat. Sb. 33 (1953), 559-566.

[16] J. Rivat, P. Sargos, *Nombres premiers de la forme* $[n^c]$, Can. J. Math. 53 (2001), 414-433.

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