

The large sieve with quadratic amplitude

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10.02.06

Abstract: We establish a large sieve bound for expressions of the form

$$\sum_{r=1}^R \left| \sum_{M < n \leq M+N} a_n e(\alpha_r f(n)) \right|^2,$$

where $f(x) = \alpha x^2 + \beta x + \theta \in \mathbb{R}[x]$ is a quadratic polynomial with $\alpha > 0$ and $\beta \geq 0$. We also consider the case when $f(x) = x^d$ with $d \in \mathbb{N}$, $d \geq 3$.

Mathematics Subject Classification (2000): 11N35, 11L07

Key words: large sieve, quadratic amplitude, double large sieve, exponential sums

1 Introduction

Throughout this paper, we suppose that Q, R, M, N are integers with $Q \geq 1$, $R \geq 1$, $N \geq 1$ and $M \geq 0$. As usual, by ε we denote a fixed but arbitrary (small) positive real number. Further, we suppose that (a_n) and (α_r) are sequences of complex numbers. We set

$$S(\alpha) := \sum_{M < n \leq M+N} a_n e(\alpha n)$$

and

$$Z := \int_0^1 |S(\alpha)|^2 d\alpha = \sum_{M < n \leq M+N} |a_n|^2.$$

By $\|x\|$ we denote the distance of a real number x to its closest integer.

In its modern form, the large sieve is an inequality connecting a discrete and the continuous mean value Z of the trigometrical polynomial $S(\alpha)$, *i.e.* an inequality of the form

$$(1) \quad \sum_{r=1}^R |S(\alpha_r)|^2 \leq \Delta(N; \alpha_1, \dots, \alpha_r) Z.$$

Montgomery and Vaughan [9] proved that (1) holds with

$$\Delta(N; \alpha_1, \dots, \alpha_r) = N + \delta^{-1},$$

where

$$(2) \quad \delta := \min_{\substack{r, s \leq R \\ r \neq s}} \|\alpha_r - \alpha_s\|.$$

In many applications, the sequence $\alpha_1, \dots, \alpha_R$ consists of Farey fractions. If $\alpha_1, \dots, \alpha_R$ is the sequence of all fractions a/q with $1 \leq a \leq q$, $(a, q) = 1$ and $q \leq Q$, then the above results implies that

$$\sum_{q \leq Q} \sum_{\substack{a=1 \\ (a, q)=1}}^q \left| S\left(\frac{a}{q}\right) \right|^2 \leq (N + Q^2) Z,$$

which is a sharpened version of the classical large sieve inequality of Bombieri [2].

In [11] L. Zhao dealt with sums of the form

$$\sum_{q \leq Q} \sum_{\substack{a=1 \\ (a, q)=1}}^q \left| \sum_{M < n \leq M+N} a_n e\left(\frac{af(n)}{q}\right) \right|^2,$$

where

$$f(x) = \alpha x^2 + \beta x + \theta \in \mathbb{R}[x]$$

is a *quadratic* polynomial with $\alpha \neq 0$. Without loss of generality, we can assume that $\alpha > 0$ (if $\alpha < 0$, then we just need to replace $f(x)$ by $-f(x)$), which we suppose from now on.

For the case when β/α is rational, Zhao established the following bound (Theorem 2. in [11]): If $\beta/\alpha = u/v$ with $u, v \in \mathbb{Z}$, $v > 0$ and $(u, v) = 1$, then

$$(3) \quad \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left| \sum_{M < n \leq M+N} a_n e\left(\frac{af(n)}{q}\right) \right|^2 \\ \ll \left(Q^2 + Q\sqrt{\alpha N(M+N) + 1} \right) \Pi Z,$$

where

$$\Pi = \left(\frac{v}{\alpha} + 1 \right)^{1/2+\varepsilon} [Nv(M+N) + |u| + v/\alpha]^\varepsilon.$$

We recall that we here suppose $M \geq 0$.

Zhao also dealt with the case when β/α is a general real number (see Proposition 1. in [11]). However, for irrational β/α his result is weaker than (3) unless β/α is in a sense well-approximable by rational numbers.

In many applications, the quantity

$$(4) \quad Z^* := N \max_{M < n \leq M+N} |a_n|^2$$

does not exceed the quantity $Z = \sum_{M < n \leq M+N} |a_n|^2$ much. In the present paper we are concerned with large sieve inequalities of the form

$$\sum_{r=1}^R \left| \sum_{M < n \leq M+N} a_n e(\alpha_r f(n)) \right|^2 \ll \Delta(M, N; \alpha_1, \dots, \alpha_r) Z^*.$$

To avoid technical complications, we confine ourselves to the case when $\beta \geq 0$. Though, our method should lead to the same result for $\beta < 0$. We shall prove

Theorem 1: *Define δ as in (2) and Z^* as in (4). Let $f(x) = \alpha x^2 + \beta x + \theta \in \mathbb{R}[x]$, where $\alpha > 0$ and $\beta \geq 0$. Then we have, with an absolute \ll -constant,*

$$(5) \quad \sum_{r=1}^R \left| \sum_{M < n \leq M+N} a_n e(\alpha_r f(n)) \right|^2 \\ \ll (1 + \alpha^{-1/2}) R^{1/2} (N^{1/2}(M+N)^{1/2} + \delta^{-1/2}) Z^* \times \\ \log^{1/2}(2 + \alpha^{-1}) \log 2N$$

if $N > N_0$, where N_0 is a non-negative constant which depends only on α and β .

An immediate consequence of Theorem 1 is

Corollary 1: Define Z^* as in (4). Let $f(x) = \alpha x^2 + \beta x + \theta \in \mathbb{R}[x]$, where $\alpha > 0$ and $\beta \geq 0$. Then we have, with an absolute \ll -constant,

$$(6) \quad \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left| \sum_{M < n \leq M+N} a_n e\left(\frac{af(n)}{q}\right) \right|^2 \\ \ll (1 + \alpha^{-1/2}) (QN^{1/2}(M+N)^{1/2} + Q^2) Z^* \times \\ \log^{1/2}(2 + \alpha^{-1}) \log 2N$$

if $N > N_0$, where N_0 is a non-negative constant which depends only on α and β .

In the following two sections we shall prove Theorem 1. In the last section we shall touch the case of polynomials $f(x)$ of degree ≥ 3 .

2 Preliminaries

Like Zhao's method in [11], our method relies on the double large sieve of Bombieri and Iwaniec (Lemma 5.2 in [1]). Here we state only the one-dimensional version of the double large sieve.

Proposition 1: Suppose that x_1, \dots, x_R and y_1, \dots, y_S are real numbers with

$$-\frac{X}{2} \leq x_r \leq \frac{X}{2}, \quad -\frac{Y}{2} \leq y_s \leq \frac{Y}{2}$$

for $r = 1, \dots, R$ and $s = 1, \dots, S$. Put $\Lambda(x) := \max(1 - |x|, 0)$. Then we have

$$(7) \quad \left| \sum_{r=1}^R \sum_{s=1}^S c_r d_s e(x_r y_s) \right|^2 \leq \left(\frac{\pi}{2}\right)^4 AB(XY + 1),$$

where

$$A := \sum_{r=1}^R \sum_{\rho=1}^R c_r c_\rho \Lambda((x_r - x_\rho)Y)$$

and

$$B := \sum_{s=1}^S \sum_{\sigma=1}^S d_s d_\sigma \Lambda((y_s - y_\sigma)X).$$

Using Proposition 1, we shall reduce the problem in question to estimating the number of solutions $k, l, u, v \in \mathbb{Z}$ of a Diophantine inequality of the form

$$(8) \quad |l(v + \gamma) - k(u + \gamma)| \leq h,$$

where h and γ are fixed real numbers, and the variables k, l, u, v lie in certain intervals. We shall employ the following bound which is essentially due to G. Harman.

Proposition 2: *Let $\gamma \in \mathbb{R}$ and $h, K, L, U, V \geq 1$ be given. Then the number of solutions $k, l, u, v \in \mathbb{Z}$ with $K \leq k \leq 2K$, $L \leq l \leq 2L$, $U \leq u \leq 2U$, $V \leq v \leq 2V$ of the inequality (8) is*

$$(9) \quad \ll (\min\{K, L\} \max\{U, V\} (1 + |\log K/L|) + (K + L)^{3/2+\varepsilon}) \times \\ h \log 2h \log 2(K + L),$$

where the implied \ll -constant depends only on ε .

G. Harman stated and used the bound (9) for $U = V$ in the proof of Lemma 3 in [4] (note that our notations differ from those in [4]). He did not prove this bound in [4] but referred to his paper [3] in which he established a similar bound, Lemma 7, for irrational real γ 's which satisfy the condition

$$(10) \quad \|q\gamma\| > A^{-q}, \quad \text{all } q \in \mathbb{N},$$

for some A . Proposition 2 can also be established by the method used to prove Lemma 7 in [3]. Instead of the estimate (5.6) in [3] one here uses the slightly weaker estimate $\ll hTl^{-1}$ (see the remark at the beginning of the proof of Lemma 8 in [3]) which is satisfied for *all* real γ . We also note that the term h^2 in (5.3) in [3] can be replaced by $h \log 2h$ (however, for the application in [3] it was sufficient to use (5.3) with h^2). The term h^2 arose from the crude estimate $1 + \log h \ll h$ at the end of the proof of Lemma 7 in [3].

We shall also need the following slightly modified version of Proposition 2, which can be established by the same method.

Proposition 3: *Let $\gamma \in \mathbb{R}$ and $h, K, L, U, Z \geq 1$ be given. Suppose that $Z \leq U$. Then the number of solutions $k, l, u, v \in \mathbb{Z}$ with $K \leq k \leq 2K$, $L \leq l \leq 2L$, $U \leq u \leq U + Z$, $U \leq v \leq U + Z$ of the inequality (8) is*

$$(11) \quad \ll (\min\{K, L\}Z(1 + |\log K/L|) + (K + L)^{3/2+\varepsilon}) \times h \log 2h \log(K + L),$$

where the implied \ll -constant depends only on ε .

3 Proof of Theorem 1

We are now ready to prove Theorem 1, our main result. As in [11], we begin with applying the double large sieve.

Multiplying out the square, we get

$$(12) \quad \begin{aligned} & \sum_{r=1}^R \left| \sum_{M < n \leq M+N} a_n e(\alpha_r f(n)) \right|^2 \\ &= \sum_{r=1}^R \sum_{M < m \leq M+N} \sum_{M < n \leq M+N} a_m \overline{a_n} e(\alpha_r (f(m) - f(n))) \\ &= \sum_{r=1}^R \sum_{M < m \leq M+N} \sum_{M < n \leq M+N} a_m \overline{a_n} e(\alpha_r \alpha (m - n)(m + n + \beta/\alpha)). \end{aligned}$$

In the remaining part of this paper, we assume without loss of generality that

$$-1/2 \leq \alpha_r \leq 1/2$$

for $r = 1, \dots, R$, and we put $\gamma := \beta/\alpha$. Then, applying Proposition 1 with

$$(x_r)_{1 \leq r \leq R} = (\alpha \alpha_r)_{1 \leq r \leq R}, \quad (y_s)_{1 \leq s \leq S} = ((m - n)(m + n + \gamma))_{M < m, n \leq M+N}, \\ (c_r) \equiv 1, \quad (d_s)_{1 \leq s \leq S} = (a_m \overline{a_n})_{M < m, n \leq M+N}, \quad X = \alpha, \quad Y = 2N(M + N + \gamma),$$

we obtain

$$(13) \quad \left| \sum_{r=1}^R \sum_{M < m \leq M+N} \sum_{M < n \leq M+N} a_m \overline{a_n} e(\alpha_r \alpha (m-n)(m+n+\gamma)) \right|^2 \\ \ll AB(\alpha N(M+N+\gamma) + 1) \max_{M < n \leq M+N} |a_n|^4,$$

where A is the number of solutions α_r, α_ρ with $1 \leq r, \rho \leq R$ of the inequality

$$|\alpha_r - \alpha_\rho| \leq \frac{1}{2\alpha N(M+N+\gamma)},$$

and B is the number of solutions $m_1, n_1, m_2, n_2 \in \mathbb{Z}$ with $M < m_1, n_1, m_2, n_2 \leq M+N$ of the inequality

$$|(m_1 - n_1)(m_1 + n_1 + \gamma) - (m_2 - n_2)(m_2 + n_2 + \gamma)| \leq 1/\alpha.$$

Since the sequence $\alpha_1, \dots, \alpha_R$ is well-spaced with spacing δ , we have

$$(14) \quad A \leq R \left(1 + \frac{1}{\delta \alpha N(M+N+\gamma)} \right).$$

Obviously, B is \leq the number B' of solutions $k, l, u, v \in \mathbb{Z}$ with

$$(15) \quad -2N \leq k, l \leq 2N, \quad 2M < u, v \leq 2(M+N)$$

of the inequality

$$(16) \quad |l(v+\gamma) - k(u+\gamma)| \leq 1 + 1/\alpha.$$

In the following, we derive an estimate for B' . We always suppose that the conditions in (15) are satisfied.

Case 1: If $k = 0$, then (16) has

$$\ll N \sum_{2M < v \leq 2(M+N)} \left(1 + \frac{1 + \alpha^{-1}}{v + \gamma} \right) \ll N^2 + N(1 + \alpha^{-1}) \log 2N$$

solutions (l, u, v) .

Case 2: Similarly, if $l = 0$, then (16) has $\ll N^2 + N(1 + \alpha^{-1}) \log 2N$ solutions (k, u, v) .

Case 3: Suppose that $k < 0$ and $l > 0$. Then a crude bound for the number of solutions k, l, u, v of (16) is

$$\ll \left(\sum_{1 \leq t \leq 1+1/\alpha} d(t) \right)^2 \ll (1 + \alpha^{-1})^2 \log^2 2(1 + \alpha^{-1}),$$

where $d(t)$ is the number of divisors of t .

Case 4: Suppose that $k > 0$ and $l < 0$. Then, like in Case 3, there are $\ll (1 + \alpha^{-1})^2 \log^2 2(1 + \alpha^{-1})$ solutions k, l, u, v of (16).

Case 5: Suppose that $k > 0$, $l > 0$ and $M \geq N$. Put $J := \lceil \log_2 N \rceil + 1$. Then, by Proposition 3, the number of solutions k, l, u, v of (16) is

(17)

$$\begin{aligned} &\ll \sum_{i=0}^J \sum_{j=0}^J \left(\min \left\{ \frac{N}{2^i}, \frac{N}{2^j} \right\} N(1 + |\log(2^j/2^i)|) + N^{3/2+\varepsilon} \right) \times \\ &\quad (1 + \alpha^{-1}) \log 2(1 + \alpha^{-1}) \log 2N \\ &\ll \left(N^{3/2+2\varepsilon} + N^2 \sum_{i=0}^J \sum_{j=0}^J \min \{ 2^{-i}, 2^{-j} \} (1 + |j - i|) \right) \times \\ &\quad (1 + \alpha^{-1}) \log 2(1 + \alpha^{-1}) \log 2N. \end{aligned}$$

The double sum in the last line of (17) can be estimated by

$$\begin{aligned}
(18) \quad & \sum_{i=0}^J \sum_{j=0}^J \min \{2^{-i}, 2^{-j}\} (1 + |j - i|) \\
& \ll \sum_{i=0}^J \sum_{j=i}^J 2^{-j} (1 + j - i) \\
& \ll \sum_{i=0}^J \sum_{j=i}^J \left(\frac{2}{3}\right)^j \\
& \ll \sum_{i=0}^J \left(\frac{2}{3}\right)^i,
\end{aligned}$$

and the sum in the last line of (18) is bounded by a constant. So the number of solutions in question is

$$\ll N^2(1 + \alpha^{-1}) \log 2(1 + \alpha^{-1}) \log 2N.$$

Case 6: Suppose that $k > 0$, $l > 0$ and $M < N$. Put $J := \lceil \log_2 N \rceil + 1$. Then, by Proposition 2, the number of solutions k, l, u, v of (16) is

$$\begin{aligned}
(19) \quad & \ll (1 + \alpha^{-1}) \log 2(1 + \alpha^{-1}) \log 2N \sum_{i=0}^J \sum_{j=0}^J \sum_{f=0}^{J+1} \sum_{g=0}^{J+1} \\
& \left(\min \left\{ \frac{N}{2^i}, \frac{N}{2^j} \right\} \max \left\{ \frac{M+N}{2^f}, \frac{M+N}{2^g} \right\} (1 + |\log(2^j/2^i)|) + N^{3/2+\varepsilon} \right).
\end{aligned}$$

In a similar manner like in Case 5 one proves that the expression in (19) is

$$\ll N^2(1 + \alpha^{-1}) \log 2(1 + \alpha^{-1}) \log^2 2N.$$

Case 7: Suppose that $k < 0$, $l < 0$ and $M \geq N$. Then we get the same bound like in Case 5.

Case 8: Suppose that $k < 0$, $l < 0$ and $M < N$. Then we get the same bound like in Case 6.

Collecting all contributions together, we find that the total number of solutions k, l, u, v of (16) is

$$(20) \quad \ll N^2(1 + \alpha^{-1}) \log 2(1 + \alpha^{-1}) \log^2 2N$$

if $N > N_0(\alpha)$, where $N_0(\alpha)$ is a non-negative constant which depends only on α .

Now, combining (12), (13), (14) and the bound (20) for the term B , we obtain the result of Theorem 1. \square

4 Polynomials of higher degree

In this section we deal with the simplest polynomials of higher degree, namely the polynomials $f(x) = x^d$ with $d \geq 3$. Our aim is to estimate the expression

$$\sum_{r=1}^R \left| \sum_{M < n \leq M+N} a_n e(\alpha_r n^d) \right|^2.$$

For simplicity, we confine ourselves to the case when $M = 0$. In what follows, we allow the implied \ll -constants to depend on d and on some parameter k which we introduce below.

Using Hölder's inequality, we get for $k \geq 2$

$$(21) \quad \sum_{r=1}^R \left| \sum_{n=1}^N a_n e(\alpha_r n^d) \right|^2 \leq R^{1-2/k} \left(\sum_{r=1}^R \left| \sum_{n=1}^N a_n e(\alpha_r n^d) \right|^k \right)^{2/k}.$$

If $k \in \mathbb{N}$, then

$$(22) \quad \begin{aligned} & \sum_{r=1}^R \left| \sum_{n=1}^N a_n e(\alpha_r n^d) \right|^k \\ &= \sum_{r=1}^R \left| \sum_{n_1=1}^N \dots \sum_{n_k=1}^N a_{n_1} \dots a_{n_k} e(\alpha_r (n_1^d + \dots + n_k^d)) \right|^k \\ &= \sum_{r=1}^R \sum_{n_1=1}^N \dots \sum_{n_k=1}^N \epsilon_r a_{n_1} \dots a_{n_k} e(\alpha_r (n_1^d + \dots + n_k^d)) \end{aligned}$$

for suitable complex ϵ_r with $|\epsilon_r| = 1$.

Applying Proposition 1 with

$$\begin{aligned} (x_r)_{1 \leq r \leq R} &= (\alpha_r)_{1 \leq r \leq R}, & (y_s)_{1 \leq s \leq S} &= (n_1^d + \dots + n_k^d)_{0 < n_1, \dots, n_k \leq N}, \\ (c_r)_{1 \leq r \leq R} &= (\epsilon_r)_{1 \leq r \leq R}, & (d_s)_{1 \leq s \leq S} &= (a_{n_1} \cdots a_{n_k})_{0 < n_1, \dots, n_k \leq N}, \\ X &= 1, & Y &= 2kN^d, \end{aligned}$$

we obtain

$$(23) \quad \left| \sum_{r=1}^R \sum_{n_1=1}^N \cdots \sum_{n_k=1}^N \epsilon_r a_{n_1} \cdots a_{n_k} e(\alpha_r (n_1^d + \dots + n_k^d)) \right|^2 \\ \ll ABN^d \max_{n \leq N} |a_n|^{2k},$$

where A is the number of solutions α_r, α_ρ with $1 \leq r, \rho \leq R$ of the inequality

$$|\alpha_r - \alpha_\rho| \leq \frac{1}{2kN^d},$$

and B is the number of solutions $(m_1, \dots, m_k, n_1, \dots, n_k) \in \mathbb{N}^{2k}$ with $m_1, \dots, m_k, n_1, \dots, n_k \leq N$ of the equation

$$m_1^d + \dots + m_k^d - (n_1^d + \dots + n_k^d) = 0.$$

Since the sequence $\alpha_1, \dots, \alpha_R$ is well-spaced with spacing δ , we have

$$(24) \quad A \leq R \left(1 + \frac{1}{\delta k N^d} \right).$$

Combining (21), (22), (23) and (24), we obtain

Theorem 2: *Define δ as in (2). Suppose that $d, k \in \mathbb{N}$, $d \geq 3$ and $k \geq 2$. Then we have*

$$(25) \quad \sum_{r=1}^R \left| \sum_{n=1}^N a_n e(\alpha_r n^d) \right|^2 \ll R^{1-1/k} (N^{d/k} + \delta^{-1/k}) B_{d,k}^{1/k}(N) \max_{n \leq N} |a_n|^2,$$

where

$$B_{d,k}(N) := |\{(m_1, \dots, m_k, n_1, \dots, n_k) \in \mathbb{N}^{2k} : m_1, \dots, m_k, n_1, \dots, n_k \leq N, \\ m_1^d + \dots + m_k^d = n_1^d + \dots + n_k^d\}|.$$

The term $B_{d,k}(N)$ can be expressed in the form

$$B_{d,k}(N) = \int_0^1 \left| \sum_{n=1}^N e(\alpha n^d) \right|^{2k} d\alpha,$$

and this integral can be estimated by using Hua's inequality (see [7]). In particular, for $d = 3 = k$ Hua's inequality yields (see [5])

$$B_{3,3}(N) \ll N^{7/2+\varepsilon}.$$

Hooley [6] and Heath-Brown [5] established independently the much sharper bound

$$B_{3,3}(N) \ll N^{3+\varepsilon}$$

under the Riemann hypothesis for certain Hasse-Weil L -functions. Thus, Theorem 2 implies

Theorem 3: *Define δ as in (2) and Z^* as in (4). Then we have*

$$(26) \quad \sum_{r=1}^R \left| \sum_{n=1}^N a_n e(\alpha_r n^3) \right|^2 \ll R^{2/3} (N + \delta^{-1/3}) N^{1/6+\varepsilon} Z^*.$$

If the Riemann hypothesis for Hasse-Weil L -functions holds true, then the left-hand side of (26) is

$$\ll R^{2/3} (N + \delta^{-1/3}) N^\varepsilon Z^*.$$

In particular, for the special case of Farey fractions we obtain

Corollary 2: *Define Z^* as in (4). Then we have*

$$(27) \quad \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left| \sum_{n=1}^N a_n e\left(\frac{an^3}{q}\right) \right|^2 \ll (Q^{4/3}N + Q^2) N^{1/6+\varepsilon} Z^*.$$

If the Riemann hypothesis for Hasse-Weil L -functions holds true, then the left-hand side of (27) is

$$\ll (Q^{4/3}N + Q^2) N^\varepsilon Z^*.$$

Heuristically, one may expect that

$$(28) \quad B_{d,k}(N) \ll N^{\max\{k, 2k-d\}+\varepsilon}.$$

If this inequality holds, then for large N the optimal choice of the parameter k in Theorem 4 is $k = d$. In this case ($k = d$) the bound (28) follows from Hooley's hypothesis K^* in Waring's problem (see [6]) which asserts that

$$\sum_{n \leq X} R_{d,d}^2(n) \ll X^{1+\varepsilon},$$

where $R_{d,d}(n)$ is the number of solutions $(n_1, \dots, n_d) \in \mathbb{N}^d$ of the equation

$$n_1^d + \dots + n_d^d = n.$$

Thus, Theorem 2 implies

Theorem 4: *Define δ as in (2) and Z^* as in (4). Let $d \geq 3$ be a natural number. Assume that hypothesis K^* holds. Then we have*

$$(29) \quad \sum_{r=1}^R \left| \sum_{n=1}^N a_n e(\alpha_r n^d) \right|^2 \ll R^{1-1/d} (N + \delta^{-1/d}) N^\varepsilon Z^*.$$

In particular, for the special case of Farey fractions we obtain

Corollary 3: *Define Z^* as in (4). Let $d \geq 3$ be a natural number. Assume that hypothesis K^* holds. Then we have*

$$(30) \quad \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left| \sum_{n=1}^N a_n e\left(\frac{an^d}{q}\right) \right|^2 \ll (Q^{2(1-1/d)} N + Q^2) N^\varepsilon Z^*.$$

Actually, Hooley [6] and Heath-Brown [5] proved the hypothesis K^* for $d = 3$ under the Riemann hypothesis for certain Hasse-Weil L -functions.

We note that for $d = 2$ the bounds (29) and (30) with $\log 2N$ in place of N^ε follow from Theorem 1 and Corollary 1. For $d = 1$ the bounds (29) and (30) with the term N^ε omitted follow from the ordinary large sieve inequalities given at the beginning of this paper.

Acknowledgement. This paper was written when the author held a post-doctoral position at the Harish-Chandra Research Institute at Allahabad (India). The author wishes to thank this institute for financial support.

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